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Function Spaces and Reproducing Kernels on Bounded Symmetric Domains

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INTRODUCTION

There are several natural Hilbert spaces of holomorphic functions on a bounded symmetric domain. Such are the Bergman-type spaces, on which the holomorphic discrete series operates, and the Hardy-type spaces which are related to the analytic continuation of the holomorphic discrete series. Also closely related is the Bergmann space of entire functions on the ambient $\mathbb{C}^n$ which arises as the closure of the polynomials with respect to a natural inner product.

The space of holomorphic polynomials decomposes into irreducible subspaces under the action of the isotropy group K of the domain. The main facts about this decomposition were proved by Schmid [21]; for another proof see [22]. Each irreducible subspace contains a unique normalized L-invariant ("spherical") polynomial, where L is the isotropy group of the Shilov boundary in K. Our first main result is the explicit computation of the norms of the spherical polynomials with respect to each of the Hilbert spaces considered. For the domains of classical type they were considered by Hua [9] and for some of the Hilbert spaces this was done before by Upmeier [24] using different methods; for certain others there are partial results in [22]. We are able to do this in a fairly simple unified way by making strong use of Gindikin's generalized Gamma function [5]. Next we obtain a description of the reproducing kernels of the K-irreducible subspaces in each of our Hilbert spaces, and an expansion in terms of these for

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all complex powers of the Bergman kernel. This generalizes the expansion found by Ørsted [17] for a special class of symmetric domains.

These results have several applications. First, we shall use them to give a generalization to all symmetric domains of some inequalities due to Forelli and Rudin [4] in the case of the complex unit ball. One can use these inequalities to generalize the construction given in [4] of families of projection operators onto L^p-spaces of holomorphic functions. Furthermore, they make it possible to extend some results of Coifman and Rochberg [3] to all bounded symmetric domains, and they also imply an estimate needed by Berger, Coburn, and Zhu [2] for their theory of Toeplitz-type operators.

Another application concerns the analytic continuation of the (scalar-valued) holomorphic discrete series. The realization of the unitarizable representations on spaces of holomorphic functions, equivalent to the result in [18], can be read off our expansion of the powers of the Bergman kernel in an almost trivial way. But this expansion also yields new results: We can decide the question of irreducibility and can determine the entire composition series and the unitarizability of the quotients for every Harish-Chandra module obtained in the analytic continuation of the holomorphic discrete series. Wallach's characterization [25] of the unitarizable modules is, of course, a corollary of this. For the special case of $SU(n, n)$ the complete result was obtained previously by Ørsted [17]. Our results show that the case of $SU(n, n)$ is not quite typical, the general case being somewhat more complicated.

The main results are in Section 3. The applications just described are in Sections 4 and 5. In Section 1 we establish notation and summarize some background material. In Section 2 we give a new proof of Schmid's decomposition result [21] and of the identification of the highest weigth vectors in the irreducible subspaces, due to Upmeier [23], which we will also need. We include this section because our proof, which is in the spirit of our subsequent arguments, is considerably shorter and, in a way, more elementary than the proofs in the literature.

While the final text of this article was being written the preprint of M. Lassalle's note [14] came to our attention. Lassalle also obtains the expansion of the powers of the Bergman kernel (essentially our Theorem 3.8); his methods seem to be rather different from ours.

1. Preliminaries and Notation

In this section we establish our notation and recall some basic facts important for the sequel. The results, except where indicated, are contained in [7, 8, 13]; they can also be found in [20].

We shall consider an irreducible bounded symmetric domain D in the standard Harish-Chandra realization. The hypothesis of irreducibility is unessential; it is made only for simplicity. Thus, $\mathfrak{g}$ will be a simple real Lie algebra with Cartan decomposition $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$, $\mathfrak{k}$ having non-trivial center. $\mathfrak{h}$ will be a Cartan subalgebra of $\mathfrak{k}$ (and hence also of $\mathfrak{g}$).

The $\mathfrak{h}^{\mathbb{C}}$-roots of $\mathfrak{g}^{\mathbb{C}}$ that are also roots of $\mathfrak{k}^{\mathbb{C}}$ are called compact roots. We denote by Φ^+ the set of positive non-compact roots. Denoting by τ the conjugation with respect to the real form $\mathfrak{k}+i\mathfrak{p}$, we consider a basis of root vectors e_α such that $\tau e_\alpha=-e_{-\alpha}$, $[e_\alpha, e_{-\alpha}]=h_\alpha$, $[h_\alpha, e_{\pm\alpha}]=\pm 2e_{\pm\alpha}$. Setting

$$\mathfrak{p}^{\pm}=\sum_{\alpha\in\Phi^+}\mathbb{C}e_{\pm\alpha}$$

we have

$$\mathfrak{g}^{\mathbb{C}}=\mathfrak{p}^-+\mathfrak{k}^{\mathbb{C}}+\mathfrak{p}^+ \tag{1.1}$$

as a vector space direct sum. On $\mathfrak{p}^+$ (even on all of $\mathfrak{g}^{\mathbb{C}}$) we have a Hermitian inner product defined by

$$(z\,|\,w)=-B(z,\tau w),$$

where B is the Killing form.

$\gamma_1, ..., \gamma_r$ will be the strongly orthogonal roots of Harish-Chandra, with the ordering $\gamma_1>\cdots>\gamma_r$. r is the real rank of $\mathfrak{g}$. We will use the abbreviated notations

$$e_j=e_{\gamma_j},\qquad h_j=h_{\gamma_j}\qquad (1\leqslant j\leqslant r)$$
$$e=\sum_1^r e_j.$$

We denote by $\mathfrak{h}^-$ the real span of the vectors ih_j, by $\mathfrak{a}^+$ the real span of the e_j, and by $\mathfrak{a}$ the real span of the vectors $i(e_j+\tau e_j)$ $(1\leqslant j\leqslant r)$. $\mathfrak{a}$ is a Cartan subalgebra of the pair $(\mathfrak{g}, \mathfrak{k})$.

$G^{\mathbb{C}}$ will be the adjointgroup of $\mathfrak{g}^{\mathbb{C}}$. G, K, $K^{\mathbb{C}}$, $P^{\pm}$, A will be the analytic subgroups corresponding to $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{k}^{\mathbb{C}}$, $\mathfrak{p}^{\pm}$, $\mathfrak{a}$. $K^{\mathbb{C}}P^-$ is a group, $G^{\mathbb{C}}/K^{\mathbb{C}}P^-$ is a compact space, and P^+ is imbedded in it as a dense open set via the inclusion $P^+\to G^{\mathbb{C}}$. Identifying $\mathfrak{p}^+$ with P^+ under the exponential map, $\mathfrak{p}^+$ is imbedded into $G^{\mathbb{C}}/K^{\mathbb{C}}P^-$. We use the notation $g\cdot z$ to indicate the action of $G^{\mathbb{C}}$ on $\mathfrak{p}^+$, when meaningful. (So $\exp(g\cdot z)\in g\exp(z)\,K^{\mathbb{C}}P^-$.)

Under this action the orbit $D=G\cdot o\simeq G/K$ is a bounded domain in $\mathfrak{p}^+$; this is the Harish-Chandra realization. D is also equal to the orbit of the unit cube of $\mathfrak{a}^+$ under K, which acts by unitary transformations. (Note that the action of K, and even of $K^{\mathbb{C}}$, coincides with the adjoint action.) The Shilov boundary of D is $S = K \cdot e \simeq K/L$.

The Cayley transform is defined by $c = \exp i(\pi/4)(e - \tau e)$. We write ${}^cG$ for cGc^{-1} and ${}^c\mathfrak{g}$ for its Lie algebra. $\mathfrak{g}_T$ (denoted $\mathfrak{g}'$ in [13]) will be the fixed point set of $\mathfrak{g}$ under $\mathrm{Ad}(c^4)$. The intersections of $\mathfrak{k}$, $\mathfrak{p}$, $\mathfrak{p}^+$ with $\mathfrak{g}_T^{\mathbb{C}}$ give corresponding decompositions $\mathfrak{g}_T = \mathfrak{k}_T + \mathfrak{p}_1$, $\mathfrak{g}_T^{\mathbb{C}} = \mathfrak{p}_1^- + \mathfrak{k}_T^{\mathbb{C}} + \mathfrak{p}_1^+$. $D_1 = D \cap \mathfrak{p}_1^+ = G_T \cdot o \simeq G_T/K_T$ is a symmetric domain in $\mathfrak{p}_1^+$ (the "tube type subdomain" of D).

$\mathfrak{n}_1^{\pm} = {}^c\mathfrak{g} \cap \mathfrak{p}_1^{\pm}$ is a real form of $\mathfrak{p}_1^{\pm}$. $\mathrm{Ad}(c^2)$ is a Cartan involution of $\mathfrak{k}_T$ (which is reductive); the corresponding decomposition is $\mathfrak{k}_T = \mathfrak{l}_T + \mathfrak{q}_1$. Writing $\mathfrak{k}_T^* = \mathfrak{l}_T + i\mathfrak{q}_1$, and K_T^* for the corresponding analytic group, the orbit $\Omega = K_T^* \cdot e \simeq K_T^*/L_T^0$ is a homogeneous selfdual cone in $\mathfrak{n}_1^+$. (L_T^0 denotes the identity component of L_T.) As described in [13] or [20], $\mathfrak{n}_1^+$ has the structure of a formally real Jordan algebra, in which Ω is the interior of the set of squares; in this article we will not need to use this structure. We recall, however, the vector space direct decomposition

$$ {}^c\mathfrak{g}_T = \mathfrak{n}_1^- + \mathfrak{k}_T^* + \mathfrak{n}_1^+ \tag{1.2} $$

and we define $\mathfrak{m}$ to be the centralizer of $i\mathfrak{h}^-$ in $\mathfrak{l}_T$; it is then also the centralizer of $i\mathfrak{h}^-$ in ${}^c\mathfrak{g}_T$ and ${}^c\mathfrak{g}$.

We have $i\mathfrak{h}^- \subset i\mathfrak{q}_1 \subset {}^c\mathfrak{g}_T$, and, since $\mathrm{Ad}(c)$ interchanges $i\mathfrak{h}^-$ with $\mathfrak{a}$, $i\mathfrak{h}^- = {}^c\mathfrak{a}$ is a Cartan subalgebra of the pair $({}^c\mathfrak{g}, {}^c\mathfrak{k})$ as well as of $({}^c\mathfrak{g}_T, {}^c\mathfrak{k}_T)$ and $(\mathfrak{k}_T^*, \mathfrak{l}_T)$. As was first shown by Moore [16], and as follows even more easily from Lemmas 11–13 of [6] together with the fact that the restricted roots form a root system, the $i\mathfrak{h}^-$-roots of ${}^c\mathfrak{g}$ are $\pm\frac{1}{2}(\gamma_j \pm \gamma_k)$, $\pm\gamma_j$, $\pm\frac{1}{2}\gamma_j$ $(1 \leqslant j, k \leqslant r)$ with respective multiplicities a, 1, and $2b$ (independent of j, k). It is easy to see that the corresponding root spaces refine the decompositions (1.1), (1.2), as

$$ \mathfrak{n}_1^+ = \sum_{j<k} \mathfrak{n}^{+jk} + \sum_j \mathfrak{n}^{+j} \tag{1.3} $$

$$ \mathfrak{p}_2^+ = \sum_j \mathfrak{p}^{+j/2} \tag{1.4} $$

$$ \mathfrak{p}^+ = \mathfrak{p}_1^+ + \mathfrak{p}_2^+, \tag{1.5} $$

where $\mathfrak{n}^{+jk}$, $\mathfrak{n}^{+j}$ are the root spaces in ${}^c\mathfrak{g}$ for $\frac{1}{2}(\gamma_j + \gamma_k)$ and γ_j, respectively, and $\mathfrak{p}^{+j/2}$ is the root space in $\mathfrak{p}^+ (\subset \mathfrak{g}^{\mathbb{C}})$ for $\frac{1}{2}\gamma_j$. Furthermore,

$$ \mathfrak{n}_K = \sum_{j<k} \mathfrak{n}_K^{jk}, \qquad \bar{\mathfrak{n}}_K = \sum_{j>k} \mathfrak{n}_K^{jk} \tag{1.6} $$

$$ \mathfrak{k}_T^* = i\mathfrak{h}^- + \mathfrak{m} + \mathfrak{n}_K + \bar{\mathfrak{n}}_K \tag{1.7} $$

$$ \mathfrak{k}^{\mathbb{C}} = (\mathfrak{k}_T^*)^{\mathbb{C}} + \sum_j \mathfrak{k}^{j/2} + \sum_j \mathfrak{k}^{-j/2}, \tag{1.8} $$

where $\mathfrak{n}_K^{jk}$ is the root space for $\frac{1}{2}(\gamma_j-\gamma_k)$ in ${}^c\mathfrak{g}$ and $\mathfrak{k}^{\pm j/2}$ the root space for $\pm\frac{1}{2}\gamma_j$ in $\mathfrak{k}^{\mathbb{C}}$. (The spaces $\mathfrak{p}^{\pm j/2}$, $\mathfrak{k}^{\pm j/2}$ do not meet ${}^c\mathfrak{g}$; the $\frac{1}{2}\gamma_j$-root space in ${}^c\mathfrak{g}$ is a real form of $\mathfrak{p}^{\pm j/2}+\mathfrak{k}^{+j/2}$.)

Writing $n=\dim_{\mathbb{C}}\mathfrak{p}^+$, $n_1=\dim_{\mathbb{C}}\mathfrak{p}_1^+$, a dimension count gives

$$n_1=\frac{r(r-1)}{2}a+r \tag{1.9}$$

$$n=n_1+rb. \tag{1.10}$$

Since we have a complete description of the root structure of $(\mathfrak{g}, \mathfrak{k})$, we can use a well-known integral formula (Theorem 5.17 in Chap. I of [8]), together with an obvious $\mathrm{Ad}(K)$-equivariant isomorphism of $\mathfrak{p}$ onto $\mathfrak{p}^+$, to obtain

$$\int_{\mathfrak{p}^+} f(z)\,dz=c\int_0^\infty\cdots\int_0^\infty\int_K f\left(k\cdot\sum_1^r t_je_j\right)dk$$
$$\times 2^r\prod t_j^{2b+1}\prod_{j<k}|t_j^2-t_k^2|^a\,dt_1\cdots dt_r. \tag{1.11}$$

Here dz is the Euclidean volume element, dk the normalized Haar measure of K, and c is a constant whose exact value will not be needed. (It can easily be computed by taking for f the characteristic function of the unit ball.) Note that when calculating $\int_D f(z)\,dz$, the integration limits ∞ have to be replaced by 1 on the right hand side.

Applying the same general integral formula and using that there is an $\mathrm{Ad}(L_T^0)$-equivariant isomorphism of $i\mathfrak{q}_1$ onto $\mathfrak{n}_1^+$ [12, Lemma 2.1] we obtain

$$\int_\Omega f(x)\,dx=c_0\int_0^\infty\cdots\int_0^\infty\int_{L_T^0} f\left(l\cdot\sum_1^r t_je_j\right)dl$$
$$\times\sum_{j<k}|t_j-t_k|^a\,dt_1\,\cdots\,dt_r. \tag{1.12}$$

Here dx is again the Euclidean volume element and c_0 is an easily computable constant whose precise value will not be needed.

2. Decomposition under K

In this section we describe the decomposition of $\mathscr{P}(\mathfrak{p}^+)$, the space of polynomials on $\mathfrak{p}^+$, into irreducible subspaces under $\mathrm{Ad}(K)$ and find the highest weight vector of each subspace. The results are due to Schmid [21] and Upmeier [23]; we formulate them in a way most convenient for our purpose and give complete proofs which are mostly based on an idea of Johnson [10] and are much shorter than the original ones.

The Koecher normfunction Δ is defined on $\mathfrak{n}_1^+$ by

$$\Delta(x) = c' \left(\int_\Omega e^{-(x \mid y)} \, dy \right)^{-r/n_1}, \tag{2.1}$$

where c' is such that $\Delta(e) = 1$. (Sometimes one finds the exponent -1 instead of $-r/n_1$, in the literature.) A variable change shows that Δ is a semi-invariant in the sense that, for $k \in K_T^*$,

$$\Delta(k \cdot x) = (\det \mathrm{Ad}_{\mathfrak{n}_1^+}(k))^{r/n_1} \, \Delta(x). \tag{2.2}$$

In particular, Δ is invariant under L_T^0 and under the entire derived group of K_T^*.

When $h \in i\mathfrak{h}^-$, $\mathrm{ad}(h)$ is diagonalized by the decomposition (1.3), and using (1.9) one finds

$$\begin{aligned} \mathrm{tr} \, \mathrm{ad}_{\mathfrak{n}_1^+}(h) &= \left[\frac{1}{2}(r-1)\, a + 1 \right] \sum_1^r \gamma_j(h) \\ &= \frac{n_1}{r} \sum_1^r \gamma_j(h). \end{aligned}$$

Setting $k = \exp h$ in (2.2), this shows that

$$\Delta(\exp(h) \cdot x) = e^{\sum \gamma_j(h)} \, \Delta(x). \tag{2.3}$$

Specializing to $x = e$, $H = \frac{1}{2} \sum_1^r (\log t_j) \, h_j$ and using strong orthogonality, we get

$$\Delta\left(\sum_1^r t_j e_j \right) = \prod_1^r t_j. \tag{2.4}$$

This shows that $\Delta|_{\mathfrak{a}^+}$ is invariant under the symmetric group, which is the Weyl group of $(\mathfrak{k}_T^*, \mathfrak{l}_T)$. By Chevalley's theorem, the L_T^0-invariant extension to $\mathfrak{p}_1^+$, i.e., Δ itself, is also a polynomial (cf. [12, Lemma 2.3]).

For $1 \leqslant q \leqslant r$, the centralizer $\mathfrak{g}^{(q)}$ of $\sum_{q+1}^r \mathbf{R} h_j$ in $\mathfrak{g}$ is reductive. For all previously introduced subspaces of $\mathfrak{g}^{\mathbb{C}}$ we will use the superscript (q) to denote the intersection with $\mathfrak{g}^{(q),\mathbb{C}}$. We then have analogues of the decompositions (1.3) to (1.8), with the same root spaces as before, but the sums on the right hand sides are over $1 \leqslant j$, $k \leqslant q$ only.

We denote by Δ_q the Koecher norm corresponding to $G^{(q)}$. It is a polynomial on $\mathfrak{p}_1^{(q)+}$, semi-invariant under $K^{(q)*}$ (note $\Delta = \Delta_r$). We extend it to a polynomial on $\mathfrak{p}^+$ by composing it with the orthogonal projection onto $\mathfrak{p}_1^{(q)+}$. For an r-tuple $\mathbf{s}=(s_1, ..., s_r)\in\mathbb{C}^r$ we write

$$\Delta_{\mathbf{s}}=\Delta_1^{s_1-s_2}\cdots\Delta_{r-1}^{s_{r-1}-s_r}\Delta_r^{s_r}. \tag{2.5}$$

When $\mathbf{s}=\mathbf{m}$ consists of integers such that $m_1\geqslant\cdots\geqslant m_r\geqslant 0$, we write $\mathbf{m}\geqslant 0$. In this case $\Delta_{\mathbf{m}}$ is a polynomial.

Let $\mathfrak{s}$ denote the subalgebra of $\mathfrak{k}^{\mathbb{C}}$ spanned by the *negative* root vectors. In the following theorem we take highest weights with respect to $\mathfrak{s}$.

2.1. THEOREM. *The space of polynomials on* $\mathfrak{p}^+$ *(resp.* $\mathfrak{p}_1^+$*) decomposes into irreducible subspaces under* $\mathrm{Ad}(K^{\mathbb{C}})$ *(resp.* $\mathrm{Ad}(K_T^{\mathbb{C}})$*) as*

$$\mathscr{P}(\mathfrak{p}^+)=\bigoplus_{\mathbf{m}\geqslant 0}\mathscr{P}_{\mathbf{m}}(\mathfrak{p}^+)$$
$$\mathscr{P}(\mathfrak{p}_1^+)=\bigoplus_{\mathbf{m}\geqslant 0}\mathscr{P}_{\mathbf{m}}(\mathfrak{p}_1^+).$$

For each $\mathbf{m}\geqslant 0$, *restriction of polynomials maps* $\mathscr{P}_{\mathbf{m}}(\mathfrak{p}^+)$ *onto* $\mathscr{P}_{\mathbf{m}}(\mathfrak{p}_1^+)$; *both of these spaces as representations of* $K^{\mathbb{C}}$ *resp.* $K_T^{\mathbb{C}}$ *have highest weight* $-(m_1\gamma_1+\cdots+m_r\gamma_r)$ *and highest weight vector* $\Delta_{\mathbf{m}}$. *For each* $\mathbf{m}\geqslant 0$, $\mathscr{P}_{\mathbf{m}}(\mathfrak{p}^+)$ *contains a unique L-invariant polynomial* $\varphi_{\mathbf{m}}$ *such that* $\varphi_{\mathbf{m}}(e)=1$. *The restriction of* $\varphi_{\mathbf{m}}$ *to* $\mathfrak{p}_1^+$ *is the unique* L_T*-invariant polynomial in* $\mathscr{P}_{\mathbf{m}}(\mathfrak{p}_1^+)$ *having value* 1 *at* e.

Proof. The polynomials of any fixed homogeneous degree form a $K^{\mathbb{C}}$-invariant space, so we are dealing with the decomposition of finite-dimensional spaces only. We write $\pi(k)\,p=p\circ\mathrm{Ad}(k^{-1})$ for every polynomial p. We denote the corresponding representation of $\mathfrak{k}^{\mathbb{C}}$ by π_*.

By the theorem of the highest weight our decomposition statements will be proved if we show that (i) $\Delta_{\mathbf{m}}$ is annihilated by $\pi_*(\mathfrak{s})$ and is an eigenfunction of $\pi_*(\mathfrak{h}^{\mathbb{C}})$ with the appropriate eigenvalues, and (ii) the only polynomials annihilated by $\pi_*(\mathfrak{s})$ and eigenfunctions of $\pi_*(\mathfrak{h}^{\mathbb{C}})$ are the $\Delta_{\mathbf{m}}$ ($\mathbf{m}\geqslant 0$).

We first consider Δ_q for fixed $1\leqslant q\leqslant r$. By (2.3) applied to the subgroup $G^{(q)}$ we immediately see that, for $h\in\mathfrak{h}^-$,

$$\pi_*(h)\,\Delta_q=-\sum_1^q\gamma_j(h)\,\Delta_q.$$

Now observe that $\mathfrak{s}+\mathfrak{h}^{\mathbb{C}}\cap\mathfrak{m}^{\mathbb{C}}\subset\bar{\mathfrak{n}}_K^{\mathbb{C}}+\sum_1^r\mathfrak{k}^{-j/2}+\mathfrak{m}^{\mathbb{C}}$, and recall (1.6). If $X\in\mathfrak{n}_K^{jk}$ with $k<j\leqslant q$, then $\pi_*(X)\,\Delta_q=0$ by the semi-invariance of Δ_q under $K^{(q)*}$. If $X\in\mathfrak{n}_K^{st}$ with $s>t$, $s>q$ or if $X\in\mathfrak{k}^{-j/2}$ for any j, then $\mathrm{ad}(X)z$ is orthogonal to $\mathfrak{p}_1^{(q)+}$ for all $z\in\mathfrak{p}_1^{(q)+}$: In fact, z is a sum of elements z' in $(\mathfrak{n}^{+jk})^{\mathbb{C}}$ and $(\mathfrak{n}^{+j})^{\mathbb{C}}$ ($j, k\leqslant q$), hence, by addition of roots, $\mathrm{ad}(X)z'$ is either zero or is in $(\mathfrak{n}^{+sk})^{\mathbb{C}}$ or $\mathfrak{p}^{+j/2}$. It follows that $\Delta_q(\exp t\,\mathrm{ad}(X)\cdot z)=\Delta_q(z)$, whence $\pi_*(X)\,\Delta_q=0$.

Writing $\bar{N}_K$ and ${}^c A$ for the analytic groups corresponding to $\bar{\mathfrak{n}}_K$ and ${}^c\mathfrak{a}=i\mathfrak{h}^-$, $K_T^*=\bar{N}_K\;{}^cAL_T^0$ is a Iwasawa decomposition; let M denote the centralizer of ${}^cA$ in L_T^0. Then, for any $m\in M$, $n\in\bar{N}_K$, $a\in{}^cA$, we have $\Delta_q(mna\cdot e)=\Delta_q(mnm^{-1})(am\cdot e)=\Delta_q(na\cdot e)$, since Δ_q is $\bar{N}_K$- invariant and $m\cdot e=e$. All points of Ω, which is open in $\mathfrak{n}_1^+$, appear in the form $na\cdot e$, so we have $\pi(m)\,\Delta_q=\Delta_q$, or $\pi(X)\,\Delta_q=0$ for $X\in\mathfrak{m}$.

This proves statement (i) for Δ_q; since $\Delta_{\mathbf{m}}$ is just a product of Δ_q's, the general form of (i) follows at once.

To prove (ii), let p be annihilated by $\pi_*(\mathfrak{s})$ and an eigenfunction of $\pi_*(\mathfrak{h}^{\mathbb{C}})$. Then in particular, for $h\in i\mathfrak{h}^-$ we have $\pi_*(h)\,p=\lambda(h)\,p$ with some linear function λ on $i\mathfrak{h}^-$, hence $\lambda=\sum_1^r s_j\gamma_j$ with some numbers s_j. By (2.3) and (2.4) then p agrees with $\Delta_{\mathbf{s}}$ on the orbit $\mathfrak{a}^+={}^cA\cdot e$. Since p and $\Delta_{\mathbf{s}}$ are both $\bar{N}_K$-invariant, they agree on Ω, and hence everywhere. Finally, $\mathbf{s}$ must consist of integers $s_1\geqslant\cdots\geqslant s_r\geqslant 0$ since each Δ_q is irreducible [12, Lemma 2.3] and since $\Delta_{\mathbf{s}}$ must be a polynomial.

We still have to prove the statements about L-invariants. It is clear that

$$\varphi_{\mathbf{m}}=\int_L \pi(l)\,\Delta_{\mathbf{m}}\,dl$$

is an L-invariant in $\mathscr{P}_{\mathbf{m}}(\mathfrak{p}^+)$ with $\varphi_{\mathbf{m}}(e)=1$. It is also clear that the restriction of any L-invariant to $\mathfrak{p}_1^+$ is L_T-invariant. In $\mathscr{P}_{\mathbf{m}}(\mathfrak{p}_1^+)$, however, there is, up to constant, only one L_T^0-invariant, since K_T^*/L_T^0 is a symmetric space. Hence, the theorem will be completely proved if we show that decomposing $\mathscr{P}_{\mathbf{m}}(\mathfrak{p}^+)$ as a K_T-module, the only L_T^0-spherical irreducible submodule is $\mathscr{P}_{\mathbf{m}}(\mathfrak{p}_1^+)$.

Suppose then that V is another such submodule. V has a highest weight vector v with a corresponding weight μ with respect to $\mathfrak{s}\cap\mathfrak{k}_T^{\mathbb{C}}$. By a theorem of Cartan and Helgason μ is zero on $\mathfrak{h}\cap\mathfrak{m}$. (If we use the result in the remark after this theorem, we have a direct verification of this fact in the present case.) By weight theory there exist negative compact roots $\alpha_1,...,\alpha_p$ such that $\pi_*(e_{\alpha_p})\cdots\pi_*(e_{\alpha_1})v=c\Delta_{\mathbf{m}}$ with $c\neq 0$. Then $\mu+\alpha_1+\cdots+\alpha_p=-\sum m_j\gamma_j$, and some of the α_j's, say $\alpha_{j_1},...,\alpha_{j_k}$, are not $\mathfrak{k}_T^{\mathbb{C}}$-roots, since $\pi_*(e_\alpha)v=0$ when α is a $\mathfrak{k}^{\mathbb{C}}$-root. Now we have $\alpha_{j_1}+\cdots+\alpha_{j_k}$ equal to a linear combination of $\mathfrak{k}_T^{\mathbb{C}}$-roots and γ_j's. To show that this is impossible we consider the element w in the center of $\mathfrak{k}^{\mathbb{C}}$ such that $\mathrm{ad}(w)\,e_\beta=e_\beta$ for all $\beta\in\Phi^+$ (then $Z=iw$ gives the complex structure of $\mathfrak{p}^+$, cf. [13]), and the element w_0 in the center of $\mathfrak{k}_T^{\mathbb{C}}$ with the analogous property with respect to $\mathfrak{g}_T$. It is easy to see [13, Sect. 3] that $w_0=\frac{1}{2}\sum_1^r h_j$ and that $w'=w-w_0$ is in $\mathfrak{h}^{\mathbb{C}}\cap\mathfrak{m}^{\mathbb{C}}$. It follows that $\gamma_j(w')=0$ $(1\leqslant j\leqslant r)$ and $\alpha(w')=0$ for all $\mathfrak{k}_T^{\mathbb{C}}$-roots, while $\alpha(w')=1$ when α is a compact root restricting to some $-\frac{1}{2}\gamma_j$, hence when $\alpha = \alpha_{j_1}, \ldots, \alpha_{j_k}$. This gives a contradiction and finishes the proof.

Remark 1. In case D is of tube type, i.e., $\mathfrak{g}_T = \mathfrak{g}$, Schmid [21] gives the decomposition of $L^2(S)$ into irreducible K-modules. This result can also be derived from our theorem. In fact, by the Stone–Weierstrass theorem the polynomials in z and in $\bar{z}$ are dense in $L^2(S)$, so it suffices to decompose the spaces $\mathscr{P}^{st}$ of the restrictions to S of the polynomials of homogeneous degree s in z and t in $\bar{z}$. By [12, Lemma 2.4] there is a polynomial map h of homogeneous degree $r-1$ such that $\bar{z} = \Delta(z)^{-1} h(z)$ for $z \in S$. Therefore, multiplication by $\Delta(z)^t$ carries $\mathscr{P}^{st}$ to $\mathscr{P}^{s+(r-1)t,0}$ while preserving irreducibility of subspaces. It follows that $L^2(S)$ is the sum of irreducible submodules $\mathscr{P}_{\mathbf{m}}$, $m_1 \geqslant \cdots \geqslant m_r$ (without the condition $m_r \geqslant 0$), having highest weight $-(m_1\gamma_1 + \cdots + m_r\gamma_r)$.

Remark 2. The dimension $d_{\mathbf{m}}$ of each $\mathscr{P}_{\mathbf{m}}(\mathfrak{p}^+)$ is explicitly computed in [24] with the aid of the Weyl dimension formula. In this article we will not need the precise value of $d_{\mathbf{m}}$.

To finish this section we note that the definition of Gindikin's Gamma function [5] can be written in our case as

$$\Gamma_\Omega(\mathbf{s}) = \int_\Omega e^{-(x|e)} \Delta_{\mathbf{s}-n_1/r}(x)\, dx \tag{2.6}$$

for all $\mathbf{s} \in \mathbb{C}^r$ such that the integral converges. (With some abuse of notation, we write $\mathbf{s} - \lambda$ for $(s_1 - \lambda, \ldots, s_r - \lambda)$.) As Gindikin [5] proves, and as one can prove somewhat more directly by parametrizing Ω by $\bar{N}_L^c A$,

$$\Gamma_\Omega(\mathbf{s}) = (2\pi)^{(n_1-r)/2} \prod_{j=1}^{r} \Gamma\left(s_j - (j-1)\frac{a}{2}\right), \tag{2.7}$$

concerning the analogue of the Beta function it follows [5] that

$$\begin{aligned} B_\Omega(\mathbf{s}, \mathbf{t}) &= \int_{\Omega \cap \{e-\Omega\}} \Delta_{\mathbf{s}-n_1/r}(x)\, \Delta_{\mathbf{t}-n_1/r}(e-x)\, dx \\ &= \frac{\Gamma_\Omega(\mathbf{s})\, \Gamma_\Omega(\mathbf{t})}{\Gamma_\Omega(\mathbf{s}+\mathbf{t})}, \end{aligned} \tag{2.8}$$

When Ω is understood we will also use for $\lambda \in \mathbb{C}$ and $\mathbf{m} \geqslant 0$ the abbreviated notation

$$(\lambda)_{\mathbf{m}} = \frac{\Gamma_\Omega(\lambda + \mathbf{m})}{\Gamma_\Omega(\lambda)}. \tag{2.9}$$

3. The Main Results

We consider the "Fischer inner product" on $\mathscr{P}(\mathfrak{p}^+)$,

$$(p \mid q)_{\mathrm{F}} = p\left(\frac{\partial}{\partial x}\right)\overline{q(x)}\,|_{x=0},$$

and denote the corresponding norm by $\|\cdot\|_{\mathrm{F}}$. It is well known, and easy to show, that this is the inner product induced on the symmetric tensors over the dual by the Euclidean structure of $\mathfrak{p}^+$. It can also be expressed in the form

$$(p \mid q)_{\mathrm{F}} = \frac{1}{\pi^n}\int_{\mathfrak{p}^+} p(z)\,\overline{q(z)}\,e^{-\|z\|^2}\,dz.$$

For any $\mathbf{m} \geqslant 0$, the finite-dimensional Hilbert space $\mathscr{P}_{\mathbf{m}}(\mathfrak{p}^+)$ with this inner product has a reproducing kernel, i.e., there exists a function $\mathbf{K}^{\mathbf{m}}$ on $\mathfrak{p}^+ \times \mathfrak{p}^+$ such that, writing $\mathbf{K}^{\mathbf{m}}_w(z) = \mathbf{K}^{\mathbf{m}}(z, w)$, for all $w \in \mathfrak{p}^+$ we have $\mathbf{K}^{\mathbf{m}}_w \in \mathscr{P}_{\mathbf{m}}(\mathfrak{p}^+)$, and

$$f(w) = (f \mid \mathbf{K}^{\mathbf{m}}_w)_{\mathrm{F}}$$

for all $f \in \mathscr{P}_{\mathbf{m}}(\mathfrak{p}^+)$.

3.1. Lemma. (a) *For all* $\mathbf{m} \geqslant 0$,

$$\mathbf{K}^{\mathbf{m}}_e = c_{\mathbf{m}}\varphi_{\mathbf{m}}$$

with

$$c_{\mathbf{m}} = \mathbf{K}^{\mathbf{m}}(e, e) = \frac{1}{\|\varphi_{\mathbf{m}}\|^2_{\mathrm{F}}}.$$

(b) *For all* $k \in K^{\mathbb{C}}$,

$$\mathbf{K}^{\mathbf{m}}_{k\cdot e} = \mathbf{K}^{\mathbf{m}}_e \circ \mathrm{Ad}_{\mathfrak{p}^+}(k)^*$$

(*the star denotes adjoint with respect to the inner product*).

Proof. To show (b), let $f \in \mathscr{P}_{\mathbf{m}}(\mathfrak{p}^+)$. Then

$$\begin{aligned}(f \mid \mathbf{K}^{\mathbf{m}}_{k\cdot e})_{\mathrm{F}} &= f(k\cdot e) = (f \circ \mathrm{Ad}(k) \mid \mathbf{K}^{\mathbf{m}}_e)_{\mathrm{F}}\\ &= (f \mid \mathbf{K}^{\mathbf{m}}_e \circ \mathrm{Ad}(k)^*)_{\mathrm{F}}.\end{aligned}$$

(The last equality follows since the inner product is induced by the inner product on $\mathfrak{p}^+$). To prove (a), note that (b) implies that $\mathbf{K}^{\mathbf{m}}_e$ is L-invariant, so $\mathbf{K}_e^{\mathbf{m}} = c_{\mathbf{m}} \varphi_{\mathbf{m}}$ with some $c_{\mathbf{m}}$, and to find $c_{\mathbf{m}}$ it suffices to evaluate both sides at e and to take the norm of both sides.

3.2. LEMMA. *For every element of* $\mathfrak{a}^+$ *of the form* $t = \sum_1^r t_j e_j$, *define* $t^2 = \sum_1^r t_j^2 e_j$. *Then*

$$\mathbf{K}^{\mathbf{m}}(t, t) = \mathbf{K}^{\mathbf{m}}(t^2, e).$$

Proof. We can write $t = a \cdot e$ with $a \in \exp i\mathfrak{h}^-$. Then, clearly, $t^2 = a^2 \cdot e$, and $\mathrm{Ad}_{\mathfrak{p}^+}(a)$ is self-adjoint, since τ is trivial on $i\mathfrak{h}^-$. So, by Lemma 3.1(b),

$$\begin{aligned} \mathbf{K}^{\mathbf{m}}(t, t) &= (\mathbf{K}_t^{\mathbf{m}} \mid \mathbf{K}_t^{\mathbf{m}})_{\mathrm{F}} = (\mathbf{K}_e^{\mathbf{m}} \circ \mathrm{Ad}(a) \mid \mathbf{K}_e^{\mathbf{m}} \circ \mathrm{Ad}(a))_{\mathrm{F}} \\ &= (\mathbf{K}_e^{\mathbf{m}} \circ \mathrm{Ad}(a)^2 \mid \mathbf{K}_e^{\mathbf{m}})_{\mathrm{F}} = \mathbf{K}^{\mathbf{m}}(t^2, e). \end{aligned}$$

Remark. t^2 is the square of t in the Jordan algebra structure of $\mathfrak{p}_1^+$. With this interpretation the lemma is true for all $t \in \mathfrak{p}_1^+$.

3.3. LEMMA. *For all* $t = \sum_1^r t_j e_j$,

$$\int_K |\varphi^{\mathbf{m}}(k \cdot t)|^2 \, dk = \frac{1}{d_{\mathbf{m}}} \varphi_{\mathbf{m}}(t^2),$$

where $d_{\mathbf{m}} = \dim \mathscr{P}_{\mathbf{m}}(\mathfrak{p}^+)$.

Proof. The reproducing property and Lemma 2.1(b) give

$$\begin{aligned} \int_K |\mathbf{K}_e^{\mathbf{m}}(k \cdot t)|^2 \, dk &= \int_K |(\mathbf{K}_k^{\mathbf{m}} \mid \mathbf{K}_e^{\mathbf{m}})_{\mathrm{F}}|^2 \, dk \\ &= \int_K |(\mathbf{K}_t^{\mathbf{m}} \mid \mathbf{K}_e^{\mathbf{m}} \circ \mathrm{Ad}(k^{-1}))_{\mathrm{F}}|^2 \, dk. \end{aligned}$$

By the Schur orthogonality relations applied to the representation space $\mathscr{P}_{\mathbf{m}}(\mathfrak{p}^+)$ of K, this equals

$$\frac{1}{d_{\mathbf{m}}} \|\mathbf{K}_t^{\mathbf{m}}\|_{\mathrm{F}}^2 \, \|\mathbf{K}_e^{\mathbf{m}}\|_{\mathrm{F}}^2.$$

Now the assertion follows from Lemmas 3.1 and 3.2.

3.4. THEOREM. *For all* $\mathbf{m} \geqslant 0$,

$$\|\varphi_{\mathbf{m}}\|_{\mathrm{F}}^2 = \frac{(n/r)_{\mathbf{m}}}{d_{\mathbf{m}}}.$$

Proof. We use (1.11), Lemma 3.3, and the variable change $t_j^2 \to s_j$ $(1 \leqslant j \leqslant r)$:

$$\frac{1}{\pi^n}\int_{\mathfrak{p}^+} e^{-\|z\|^2} |\varphi_{\mathbf{m}}(z)|^2 \, dz$$

$$= \frac{2^r c}{\pi^n}\int_0^\infty \cdots \int_0^\infty e^{-\Sigma t_j^2} \int_K |\varphi_{\mathbf{m}}(k \cdot t)|^2 \, dk$$

$$\times \prod_j t_j^{2b+1} \prod_{j<k} |t_j^2 - t_k^2|^a \, dt_1 \cdots dt_r$$

$$= \frac{c}{\pi^n d_{\mathbf{m}}} \int_0^\infty \cdots \int_0^\infty e^{-\Sigma s_j} \varphi_{\mathbf{m}}(s)$$

$$\times \prod_j s_j^b \prod_{j<k} |s_j - s_k|^a \, ds_1 \cdots ds_r.$$

Using (2.4), (1.12), and (1.10) this is seen to be equal to

$$\frac{c}{\pi^n d_{\mathbf{m}} c_0} \int_\Omega e^{-(x|e)} \varphi_{\mathbf{m}+b}(x) \, dx = \frac{c'}{d_{\mathbf{m}}} \Gamma_\Omega(m + n/r).$$

To determine the constant $c' = c/\pi^n c_0$, which is independent of $\mathbf{m}$, take $\mathbf{m} = (0, ..., 0)$. Then $d_{\mathbf{m}} = 1$ and $\varphi_{\mathbf{m}} \equiv 1$, so we obtain

$$c' = \frac{1}{\Gamma_\Omega(n/r)} \tag{3.1}$$

and the theorem follows.

Remark. In one variable this proof amounts to reducing an integral on $\mathbb{C}$ to an integral on $\mathbb{R}$ with the aid of polar coordinates. Using the Jordan algebra structure, a similar transformation is possible in the general case, and this gives another proof of the theorem.

3.5. COROLLARY. *For all $p, q \in \mathscr{P}_{\mathbf{m}}(\mathfrak{p}^+)$,*

$$\frac{(p \mid q)_{\mathrm{F}}}{(p \mid q)_S} = (n/r)_{\mathbf{m}},$$

where $(p \mid q)_S$ denotes the inner product computed in $L^2(S)$ with respect to the normalized K-invariant measure.

In fact, by Schur's lemma, the ratio of these two K-invariant inner products is constant on the irreducible space $\mathscr{P}_{\mathbf{m}}(\mathfrak{p}^+)$. So it suffices to calculate it for $p = q = \varphi_{\mathbf{m}}$. Now Lemma 3.3, with $t = e$, gives

$$\|\varphi_{\mathbf{m}}\|_S^2 = \frac{1}{d_{\mathbf{m}}} \tag{3.2}$$

and the corollary follows.

This corollary was first proved by Upmeier [24], by different methods.

In what follows we denote by p the important constant

$$p = \frac{n + n_1}{r} = (r-1)\,a + b + 2. \tag{3.3}$$

We denote by $h(z)$ the K-invariant polynomial on $\mathfrak{p}^+$ whose restriction to $\mathfrak{a}^+$ is given by

$$h\left(\sum_1^r t_j e_j\right) = \prod_{j=1}^r (1 - t_j^2). \tag{3.4}$$

It exists and is unique by Chevalley's theorem, since the small Weyl group now consists of all signed permutations of the γ_j's. (It is also clear that, interpreting t^2 as in Lemma 3.2, we have

$$h(t) = \Delta(e - t^2) \tag{3.5}$$

and that this remains true for all $t \in \mathfrak{p}_1^+$ in the Jordan algebra sense.)

Since h is real, it can be polarized. The function

$$h(z, w) = \exp \sum_1^n z_j \frac{\partial}{\partial z_j} \exp \sum_1^n \bar{w}_j \frac{\partial}{\partial \bar{z}_j} h(z) \tag{3.6}$$

is still a polynomial, holomorphic in z, and antiholomorphic in w.

We write L_λ^2 $(\lambda > p-1)$ for the Hilbert space of holomorphic functions f on D such that

$$\|f\|_\lambda^2 = c_\lambda \int_D |f(z)|^2\, h(z)^{\lambda - p}\, dz < \infty, \tag{3.7}$$

where

$$c_\lambda = \frac{1}{\pi^N} \frac{\Gamma_\Omega(\lambda)}{\Gamma_\Omega(\lambda - n/r)}. \tag{3.8}$$

(It will follow from Theorem 3.6 that $\|1\|_\lambda = 1$.) It is easy to see that $L^2_\lambda \neq \{0\}$ and that the polynomials form a dense subset in it [6, 25]. It is immediate from the mean value theorem that, for all $z \in D$, $f \mapsto f(z)$ is a continuous linear function on L^2_λ, hence a reproducing kernel, denoted $\mathbf{K}_\lambda$, exists. $\mathbf{K}_p$ is then the usual Bergman kernel.

By computing the action of A on D (cf. [1, Lemma 1.9] for such a computation for unbounded realizations of D), it is easy to see that $h(z)^{-p} = |J_g(0)|^2$, where J_g is the complex Jacobian determinant of an element $g \in G$ such that $g \cdot 0 = z$. Since $f(z) \mapsto f(g \cdot z)\, J_g(z)^{\lambda/p}$ is clearly a unitary transformation of L^2_λ (for fixed g we can take here any branch of the power function), it follows at once that

$$\mathbf{K}_\lambda(z, w) = h(z, w)^{-\lambda} \qquad (\lambda > p - 1). \tag{3.9}$$

Implicitly this equality is contained in [11, Lemma 5.7]. Another way to arrive at it is given in [20, Chap. II, Sect. 5]; it is easy to see that the expressions for $\mathbf{K}_\lambda(z, w)$ obtained there are equal to our $h(z, w)^{-\lambda}$. Yet another expression can be given in terms of the generic norm N of the associated Jordan triple system: One has $h(z, w) = N(z, \bar{w})$ with the definition of N given in [15, Sect. 4].

3.6. THEOREM. *For all $\lambda > p - 1$ and $\mathbf{m} \geq 0$,*

$$\|\varphi_{\mathbf{m}}\|^2_\lambda = \frac{1}{d_{\mathbf{m}}} \frac{(n/r)\mathbf{m}}{(\lambda)\mathbf{m}}.$$

Proof. We use (1.11), Lemma 3.3, the variable change $t_j^2 \to s_j$, and (1.12) as in the proof of Theorem 3.4 to obtain

$$\begin{aligned}
\|\varphi_{\mathbf{m}}\|^2_\lambda &= c_\lambda c \int_0^1 \cdots \int_0^1 \int_K \|\varphi_{\mathbf{m}}(kt)\|^2\, dk \\
&\quad \times \prod_1^r (1 - t_j^2)^{\lambda - p} \prod_1^r t_j^{2b+1} \prod_{i<j} |t_i^2 - t_j^2|^a\, dt_1 \cdots dt_r \\
&= \frac{c_\lambda c}{d_{\mathbf{m}}} \int_0^1 \cdots \int_0^1 \varphi_{\mathbf{m}}(s) \prod_1^r s_j^b \\
&\quad \times \prod_1^r (1 - s_j)^{\lambda - p} \prod_{i<j} |s_i - s_j|^a\, ds_1 \cdots ds_r \\
&= \frac{c_\lambda c}{c_0 d_{\mathbf{m}}} \int_{\Omega \cap \{e - \Omega\}} \varphi_{\mathbf{m}+b}(x)\, \Delta(e - x)^{\lambda - p}\, dx.
\end{aligned}$$

Since $\varphi_{\mathbf{m}+b}$ is the average of $\Delta_{\mathbf{m}+b}$ over L_T and dx is invariant under L_T, this is equal to

$$\frac{c_\lambda c}{c_0 d_{\mathbf{m}}} \int_{\Omega \cap \{e-\Omega\}} \Delta_{\mathbf{m}+b}(x)\, \Delta(e-x)^{\lambda-p}\, dx$$
$$= \frac{c_\lambda c}{c_0 d_{\mathbf{m}}} B_\Omega(\mathbf{m}+b+n_1/r, \lambda-p+n_1/r).$$

Using (1.10), (2.8), and (3.1) we obtain the theorem.

3.7. Corollary. *For all $p, q \in \mathscr{P}_{\mathbf{m}}(\mathfrak{p}^+)$, $\lambda > p-1$, $\mathbf{m} \geqslant 0$,*

$$\frac{(p \mid q)_{\mathrm{F}}}{(p \mid q)_\lambda} = (\lambda)_{\mathbf{m}}.$$

3.8. Theorem. *For all $\lambda \in \mathbb{C}$ and all $z, w \in D$, we have*

$$h(z, w)^{-\lambda} = \sum_{\mathbf{m} \geqslant 0} (\lambda)_{\mathbf{m}}\, \mathbf{K}^{\mathbf{m}}(z, w).$$

The series converges uniformly and absolutely on compact subsets of $D \times \bar{D}$.

Proof. For $\lambda > p-1$ this is only the standard expansion of the reproducing kernel of L^2_λ into an orthonormal system. In fact, if for each $\mathbf{m}$, $\{\psi^i_{\mathbf{m}}\}$ $(1 \leqslant i \leqslant d_{\mathbf{m}})$ is orthonormal for the Fischer norm, then $\mathbf{K}^{\mathbf{m}}(z, w) = \sum_i \psi^i_{\mathbf{m}}(z)\, \overline{\psi^i_{\mathbf{m}}(w)}$, and by Corollary 3.7 the system $\{(\lambda)^{1/2}_{\mathbf{m}} \psi^i_{\mathbf{m}}\}$ is orthonormal in L^2_λ.

To prove the theorem for arbitrary complex λ, we note that the left hand side is a holomorphic function of z, $\bar{w}$, and λ. This is so because $h(z, w)$ is nowhere zero because $\mathbf{K}_p(z, w) = \mathbf{K}_p(0, u)\, J_g(0)\, \overline{J_g(u)}$ with $g \in G$ such that $g \cdot 0 = z$ and writing $u = g^{-1} \cdot w$, furthermore, $\mathbf{K}_p(0, u) = 1$ by (3.9) and by the case $\lambda = p$ of our statement already proved.

The right hand side is a sum of polynomials in z, $\bar{w}$, and λ. We show that for every $\lambda_0 > p-1$ and $0 < \rho < 1$ it is majorized by a convergent series whenever $|\lambda| \leqslant \lambda_0$, $z \in \rho\bar{D}$, $w \in \bar{D}$. By analytic continuation, this will finish the proof.

We have $z = \rho k \cdot t$ with some $k \in K$, $t = \sum_1^r t_j e_j$, $0 \leqslant t_j \leqslant 1$. We use K-invariance of the inner product and homogeneity, writing $|\mathbf{m}| = \sum m_j$, to get $\mathbf{K}^{\mathbf{m}}(z, z) = \mathbf{K}^{\mathbf{m}}(\rho t, \rho t) = \rho^{2|\mathbf{m}|} \mathbf{K}^{\mathbf{m}}(t, t)$. By Lemma 3.2 this equals $\rho^{2|\mathbf{m}|} \mathbf{K}^{\mathbf{m}}_e(t^2)$; now $\mathbf{K}_e$ is holomorphic, hence assumes its maximum modulus at some point $k' \cdot e$ of the Shilov boundary. It follows, using also the Schwarz inequality, that

$$\begin{aligned}
\mathbf{K}^{\mathbf{m}}(z, z) &\leqslant \rho^{2|\mathbf{m}|} |\mathbf{K}^{\mathbf{m}}_e(k' \cdot e)| \leqslant \rho^{2|\mathbf{m}|} |(\mathbf{K}^{\mathbf{m}}_e \mid \mathbf{K}^{\mathbf{m}}_{k' \cdot e})_{\mathrm{F}}| \\
&\leqslant \rho^{2|\mathbf{m}|} \mathbf{K}^{\mathbf{m}}(e, e)^{1/2}\, \mathbf{K}^{\mathbf{m}}(k' \cdot e, k' \cdot e)^{1/2} \\
&= \rho^{2|\mathbf{m}|} \mathbf{K}^{\mathbf{m}}(e, e).
\end{aligned}$$

Similarly, we have $\mathbf{K}^{\mathbf{m}}(w, w) \leqslant \mathbf{K}^{\mathbf{m}}(e, e)$, and finally, again using the Schwarz inequality and homogeneity,

$$\begin{aligned}|\mathbf{K}^{\mathbf{m}}(z, w)| &\leqslant \mathbf{K}^{\mathbf{m}}(z, z)^{1/2}\, \mathbf{K}^{\mathbf{m}}(w, w)^{1/2} \\ &\leqslant \rho^{|\mathbf{m}|}\mathbf{K}(e, e) = \mathbf{K}^{\mathbf{m}}(\rho^{1/2}, \rho^{1/2}e).\end{aligned}$$

We note next that $|\lambda| \leqslant \lambda_0$ and $\lambda_0 > p-1$ imply $|(\lambda)_{\mathbf{m}}| \leqslant (\lambda_0)_{\mathbf{m}}$. It follows that our series is majorized by

$$\sum_{\mathbf{m} \geqslant 0} (\lambda_0)_{\mathbf{m}}\, \mathbf{K}^{\mathbf{m}}(\rho^{1/2}e, \rho^{1/2}e),$$

which is convergent by the first part of the proof.

Remark 1. Corollary 3.5 and Theorem 3.8 imply immediately that the Szegö kernel of D (i.e., the reproducing kernel of the closure of the polynomials in the $L^2(S)$-norm) is $h(z, w)^{-n/r}$. This result is known [11, Proposition 5.7], but its original proof uses Fourier transformation on the Cayley transform of D, and is therefore less straightforward.

Remark 2. It is possible to prove this expansion in a different way by proving first

$$\Delta(e-x)^{-\lambda} = \sum_{\mathbf{m} \geqslant 0} d_{\mathbf{m}} \frac{(\lambda)_{\mathbf{m}}}{(n/r)_{\mathbf{m}}} \varphi_{\mathbf{m}}(x)$$

for x in D. One starts from the formula

$$\Delta(e-x)^{-\lambda} = \frac{1}{\Gamma_\Omega(\lambda)} \int_\Omega e^{-(e-x \mid y)}\, \Delta(y)^{\lambda - n_1/r}\, dy,$$

which follows easily from (2.6) and from the semi-invariance of Δ. Differentiating under the integral sing one obtains

$$\varphi_{\mathbf{m}}\left(\frac{\partial}{\partial x}\right) \Delta(e-x)^{-\lambda}_{|x=0} = (\lambda)_{\mathbf{m}}.$$

Then the previous expansion is just the Taylor expansion of

$$\Delta(e-x)^{-\lambda}$$

at 0 which converges in D.

4. The Generalized Forelli–Rudin Inequalities

As an application of our results in Section 3 we prove the general versions of some inequalities proved by Forelli and Rudin for the unit ball [4; 19, pp. 17–19]. For $\gamma \in \mathbb{R}$, $\lambda > p-1$, and $z \in D$ we define

$$I_\gamma(z) = \int_S |h(z, u)|^{-(n/r+\gamma)}\, du \tag{4.1}$$

$$J_{\gamma,\lambda}(z) = \int_D |h(z, w)|^{-(\lambda+\gamma)}\, h(w, w)^{\lambda-p}\, dw. \tag{4.2}$$

(The integral on S is with respect to the normalized K-invariant measure.)

4.1. Theorem. *$I_\gamma(z)$ and $J_{\gamma,\lambda}$ are bounded for $z \in D$ if and only if $\gamma < -(r-1)(a/2)$. If $\gamma > (r-1)(a/2)$, we have*

$$L_\gamma(z) \approx J_{\gamma,\lambda}(z) \approx h(z, z)^{-\gamma}$$

($\approx$ means that the quotient, as a function of z, stays between two positive constants).

Proof. For a moment we write $\mu = \frac{1}{2}(\lambda+\gamma)$. We use Theorem 3.8, then the fact that $\mathbf{K}_z^{\mathbf{m}}$ and $\mathbf{K}_w^{\mathbf{m}'}$ are orthogonal for $\mathbf{m} \neq \mathbf{m}'$, and finally Corollary 3.7 to get

$$\begin{aligned} J_{\gamma,\lambda}(z) &= \int_D |h(z, w)^{-\mu}|^2\, h(w, w)^{\lambda-p}\, dw \\ &= \int_D \Big|\sum_{\mathbf{m}\geqslant 0} (\mu)_{\mathbf{m}} \mathbf{K}^{\mathbf{m}}(z, w)\Big|^2\, h(w, w)^{\lambda-p}\, dw \\ &= \sum_{\mathbf{m}\geqslant 0} |(\mu)_{\mathbf{m}}|^2\, \|\mathbf{K}_z^{\mathbf{m}}\|_\lambda^2 \\ &= \sum_{\mathbf{m}\geqslant 0} \frac{|(\mu)_{\mathbf{m}}|^2}{(\lambda)_{\mathbf{m}}}\, \mathbf{K}^{\mathbf{m}}(z, z). \end{aligned}$$

By a similar proof the same formula, with n/r in place of λ, holds for $I_\gamma(z)$. By Stirling's formula, if α, β are not poles of Γ_Ω, we have

$$\left\|\frac{(\alpha)_{\mathbf{m}}}{(\beta)_{\mathbf{m}}}\right\| \approx \prod_{j=1}^{r} (m_j+1)^{\alpha-\beta} \tag{4.3}$$

as $\mathbf{m}$ varies. By Theorem 3.8, this implies the statements about the case $\gamma > (r-1)(a/2)$.

It is clear from the definition that if $J_{\gamma,\lambda}$ is bounded for $\gamma < \gamma_0$ then it is also bounded for every $\gamma < \gamma_0$. Therefore in discussing the boundedness question it is enough to consider values of γ near $-(r-1)(a/2)$.

It is also clear that $J_{\gamma,\lambda}(z)$ is bounded if and only if $J_{\gamma,\lambda}(e) < \infty$, i.e., by Lemma 3.1 and Theorem 3.4, if and only if

$$\sum_{\mathbf{m} \geqslant 0} d_{\mathbf{m}} \frac{|(\mu)_{\mathbf{m}}|^2}{(\lambda)_{\mathbf{m}}(n/r)_{\mathbf{m}}} < \infty. \tag{4.4}$$

When λ is large enough, this sum is equal to

$$\frac{\Gamma_\Omega(\lambda)}{\Gamma_\Omega(\mu)\,\Gamma_\Omega(\lambda-\mu)} \int_{\Omega \cap (e-\Omega)} \Delta(x)^{\mu - n/r}\,\Delta(e-x)^{\lambda-\mu-n/r}\,\Delta(e-x)^{-\mu}\,dx,$$

as we see by substituting the expansion of Theorem 3.8 to the last factor under the integral sign and then using Theorem 3.6.

By (2.8) the integral converges if and only if $\gamma < -(r-1)(a/2)$. This proves the assertion for large λ. But (4.3) shows that the convergence of (4.4) is independent of the value of λ, finishing the proof.

Remark 1. In the first manuscript of this paper the statement of this theorem contained an error which was pointed out to us by Mr. Zhimin Yan. In an article in preparation Mr. Yan will discuss the asymptotics of I_γ and $J_{\gamma,\lambda}$ when $|\gamma| \leqslant (r-1)(a/2)$.

Remark 2. Since $P(z,u) = h(z,z)^{n/r}\,|h(z,u)|^{-2n/r}$ $(z \in D,\ u \in S)$ is the Poisson kernel of D (cf. [11]), the statement about $I_\gamma(z)$ gives the asymptotic behaviour of certain spherical functions in the degenerate principal series corresponding to the Shilov boundary.

Remark 3. The further results of Forelli and Rudin [4] (cf. also [19, pp. 120–125]) can be generalized as follows. Since, for real numbers $\lambda > p-1$, $h(z,w)^{-\lambda}$ is the reproducing kernel of L^2_λ, it is clear that the operator $\tilde{T}_\lambda$ defined by

$$\tilde{T}_\lambda f(z) = c_\lambda \int_D h(z,w)^{-\lambda}\,h(w)^{\lambda-p} f(w)\,dw \tag{4.5}$$

is the orthogonal projection of the L^2-space of D with respect to the measure $c_\lambda h(w)^{\lambda-p}\,dw$ onto L^2_λ, the subspace of holomorphic functions. (When D is the unit ball in $\mathbb{C}^n$, $\tilde{T}_{\lambda+p}$ agrees with the T_λ of [4, 19].) Now, the expression (4.3) makes sense for complex values of λ as well (by (3.8), c_λ is a meromorphic function of λ). Furthermore, it can be regarded as an operator on the L^p-spaces of D with respect to Euclidean measure. When $\operatorname{Re}\lambda > p-1$, the right hand side of (4.5) with $f(w) \equiv 1$ is convergent, and gives $\tilde{T}_\lambda 1 = 1$: This is shown by our proof of Theorem 3.6, which is valid without change for complex λ. More generally, it follows by analytic continuation from the case of real λ that $\tilde{T}_\lambda f = f$ for holomorphic f, when $\tilde{T}_\lambda f$ is meaningful. As in [4], one can then show that the $\tilde{T}_\lambda$'s, for appropriately restricted complex values of λ, form families of bounded projection operators of the L^p-spaces onto their subspaces spanned by the holomorphic functions.

As in the case of the ball, the case $\lambda = 2p$ is particularly simple and natural: It is clear from (3.9) and from the reproducing property of the Bergman kernel $\mathbf{K}_p$ that

$$\int_D |h(z, w)^{-2p} h(w)^p| \, dz = h(w)^p \frac{1}{c_p} (\mathbf{K}_{p,w} \mid \mathbf{K}_{p,w})_p = \frac{1}{c_p}.$$

Therefore,

$$\int_D |\tilde{T}_{2p} f(z)| \, dz \leqslant \frac{c_{2p}}{c_p} \int_D |f(z)| \, dz$$

and it follows that $\tilde{T}_{2p}$ is a bounded projection of L^1 onto its subspace of holomorphic functions.

Remark 4. The case $\gamma > (r-1)(a/2)$ of the theorem implies a version of Lemma 2.2 in [3] valid for all bounded symmetric domains. It can be shown that this lemma in the form it is stated fails for the parameter values corresponding to $0 < \gamma < (r-1)(a/2)$. When $\gamma < -(r-1)(a/2)$ the theorem gives an inequality needed by Berger, Coburn, and Zhu in their work [2] on Toeplitz operators.

5. Analytic Continuation of the Holomorphic Discrete Series

In this section we will describe the full composition series of the Harish-Chandra modules obtained by analytic continuation of the scalar-valued holomorphic discrete series of representations of $\tilde{G}$, the universal covering group of G. For the case of $SU(n, n)$ this was done previously by Ørsted [17]. Our result also includes the discussion of unitarizability for all the irreducible modules obtained, so it implies the corresponding result of Wallach [25].

Before actually doing this, we follow Rossi and Vergne in making the following remarks. It is clear from (3.9) and the remarks preceding it that, for $\lambda > p-1$,

$$(U_\lambda(g) f)(z) = f(g^{-1} \cdot z) \, J_{g^{-1}}(z)^{\lambda/p} \tag{5.1}$$

defines a unitary representation of $\tilde{G}$ on L^2_λ. (The powers of $J_g(z)$ can be consistently defined since $\tilde{G} \times D$ is simply connected.) This statement is equivalent with the property $h(z, w) = J_g(z)^{1/p}\, h(g \cdot z, g \cdot w)\, \overline{J_g(w)}^{1/p}$; raising this equality to some power $-\lambda$, it is immediately clear that, provided $h(z, w)^{-\lambda}$ is positive definite on $D \times D$, (5.1) still defines a unitary representation on the Hilbert space of holomorphic functions determined by $h(z, w)^{-\lambda}$. The following general lemma permits us to describe the values of λ that give such representations; Rossi and Vergne [18] obtained an equivalent result by using the halfplane type realization of D and the Laplace transform.

5.1. LEMMA. *Let H be a Hilbert space of functions with a reproducing kernel $\mathbf{K}_z$. Assume H is the orthogonal direct sum of subspaces H_m. Let $\mathbf{K}^m_z = P_m \mathbf{K}_z$, where P_m is the orthogonal projection onto H_m. Let $\{b_m\}$ be a bounded set of numbers and let $\mathbf{K}^{(b)}_z = \sum b_m \mathbf{K}^m_z$, for all z. Then $\mathbf{K}^{(b)}$ is positive definite if and only if $b_m \geqslant 0$ for all m.*

Proof. $K^{(b)}_z$ is well defined since the sum defining it converges in H and hence pointwise. Now note that the finite linear combinations $\sum a_j \mathbf{K}_{z_j}$ are dense in H. (In fact, $f \perp \mathbf{K}_z$ for all z implies $f(z) = (f \mid \mathbf{K}_z) = 0$, so $f = 0$.) We have

$$\sum_{j,k} \mathbf{K}^{(b)}(z_k, z_j)\, a_j \overline{a_k} = \sum_m b_m \left(\sum_j a_j \mathbf{K}^m z_j \,\middle|\, \sum_j a_j \mathbf{K}^m z_j \right)$$

and it is clear that this is non-negative for all z_j and a_j if and only if $b_m \geqslant 0$ for all m.

5.2. COROLLARY. *$h(z, w)^{-\lambda}$ is positive definite on $D \times D$ if and only if $\lambda > (a/2)(r-1)$ or $\lambda = (a/2)\, j$ with some $0 \leqslant j \leqslant r-1$.*

Proof. It is easy to see from the explicit formula for $(\lambda)_{\mathbf{m}}$ that $(\lambda)_{\mathbf{m}} \geqslant 0$ for all $\mathbf{m} \geqslant 0$ exactly for these values of λ (for more detail cf. the proof of Theorem 5.4). The corollary follows by applying the lemma to $H = L^2_{\lambda_0}$ with sufficiently large positive λ_0.

If $\lambda > (a/2)(r-1)$, then, for all $\mathbf{m}$, $(\lambda)_{\mathbf{m}} > 0$. The corresponding Hilbert space is

$$\bigoplus_{\mathbf{m} \geqslant 0} \mathscr{P}_{\mathbf{m}}.$$

If $\lambda = (a/2)\, j$, $0 \leqslant j \leqslant r-1$, then, for $\mathbf{m} = (m_1, ..., m_j, 0, ..., 0)$, $(\lambda)_{\mathbf{m}} > 0$, and for all other $\mathbf{m}$, $(\lambda)_{\mathbf{m}} = 0$. The corresponding Hilbert space is

$$\bigoplus_{\substack{\mathbf{m} \geqslant 0 \\ m_{j+1} = \cdots = m_r = 0}} \mathscr{P}_{\mathbf{m}}.$$

Remark. For $\lambda=(a/2)(r-1)$, using an argument similar to the one in Remark 2 after Theorem 3.8, it is possible to reprove the result of Upmeier [24] stating that the sum

$$\sum_{\mathbf{m}\geqslant 0,\, m_r=0} \mathscr{P}_{\mathbf{m}}$$

is the space of harmonic polynomials, in the sense of

$$\Delta\left(\frac{\partial}{\partial z}\right) p(z)=0.$$

Now we proceed, independently of this corollary, to discuss our family of Harish-Chandra modules. First we introduce some definitions and notations, closely following Ørsted [17].

For $\lambda>p-1$, we denote by u_λ the representation of $\mathfrak{g}$ induced by U_λ. Thus, for $X\in\mathfrak{g}$,

$$(u_\lambda(X)\,f)(z)=-(Xf)(z)-\frac{\lambda}{p}\,j_X(z)\,f(z), \tag{5.2}$$

where $(Xf)(z)=(d/dt)_0 f(\exp tX\cdot z)$ as usual, and

$$j_X(z)=\frac{d}{dt}\bigg|_0 J_{\exp tX}(z). \tag{5.3}$$

It is easy to see with the aid of the Campbell–Hausdorff formula that, identifying $\mathfrak{p}^+$ with its tangent space, $X\in\mathfrak{p}^-$ acts by the quadratic vector field $\frac{1}{2}[[z,X],z]$.

The set of k-finite vectors in L^2_λ is just $\mathscr{P}=\mathscr{P}(\mathfrak{p}^+)$, and u_λ together with the action of $\tilde{k}$ via U_λ gives $\mathscr{P}$ the structure of a Harish-Chandra module. (Note that, for $k\in\tilde{k}$, $J_k(z)=\det \mathrm{Ad}_{\mathfrak{p}^+}(k)$ is a character of $\tilde{k}$, independent of z.) Comparing two different values of λ, we see that $j_X(z)$ is a polynomial in z. It follows at once that (5.2) makes sense for every $\lambda\in\mathbb{C}$ and that the U_λ-action of $\tilde{k}$ also extends to every $\lambda\in\mathbb{C}$, still defining on $\mathscr{P}$ the structure of a Harish-Chandra module. For brevity we denote by $\mathscr{P}^{(\lambda)}$ the set $\mathscr{P}$ equipped with this structure.

For every $\lambda\in\mathbb{C}$ we denote by $(\cdot\,|\,\cdot)$ the Hermitian form on $\mathscr{P}$ determined by

$$(\varphi^l_{\mathbf{m}}\,|\,\varphi^{l'}_{\mathbf{m}'})_\lambda=(\lambda)^{-1}_{\mathbf{m}}\,\delta_{\mathbf{m},\mathbf{m}'}\,\delta_{l,l'}, \tag{5.4}$$

where, for all $\mathbf{m}\geqslant 0$, $\{\varphi^l_{\mathbf{m}}\}$ $(1\leqslant l\leqslant d_{\mathbf{m}})$ is an orthonormal basis of $\mathscr{P}_{\mathbf{m}}$ for the Fischer norm. This form is invariant, i.e.,

$$(u_\lambda(X)\,p\,|\,q)_\lambda=-(p\,|\,u_\lambda(X)q)_\lambda \tag{5.5}$$

for all $X \in \mathfrak{g}^{\mathbb{C}}$, $p, q \in \mathscr{P}$. In fact the two sides are equal as meromorphic functions of λ. This follows by analytic continuation from the case $\lambda > p-1$, where (5.5) expresses the unitarity of U_λ.

In the following we consider $\mathscr{P}^{(\lambda)}$ for fixed $\lambda \in \mathbb{C}$ and make the following definitions. We denote by q the largest possible multiplicity of the number λ as a zero of the functions $\lambda' \mapsto (\lambda')_{\mathbf{m}}$ ($\mathbf{m} \geqslant 0$).

For $0 \leqslant j \leqslant q$ we define

$$M_j = \bigoplus \mathscr{P}_{\mathbf{m}}$$

with the sum over those $\mathbf{m}$ for which λ is a zero with multiplicity at most j of $\lambda' \mapsto (\lambda')_{\mathbf{m}}$.

5.3. THEOREM. *For any fixed $\lambda \in \mathbb{C}$, q is equal to the number of non-positive integers among λ, $\lambda - a/2, ..., \lambda - (r-1)\, a/2$. In particular, $q > 0$ if and only if $\lambda - (r-1)\, a/2$ or $\lambda - (r-2)\, a/2$ is a non-positive integer. In any case,*

$$M_0 \subset M_1 \subset \cdots \subset M_q = \mathscr{P}$$

is a composition series of $\mathscr{P}^{(\lambda)}$. For every $1 \leqslant j \leqslant q$, M_j/M_{j-1} has an invariant Hermitian form given by

$$(\varphi^l_{\mathbf{m}} \mid \varphi^{l'}_{\mathbf{m}'})_{\lambda, j} = \lim_{\lambda' \to \lambda} \frac{(\lambda' - \lambda)^j}{(\lambda')_{\mathbf{m}}}\, \delta_{\mathbf{m},\mathbf{m}'}\, \delta_{l,l'}. \tag{5.6}$$

Proof. By (2.7) and (2.9), with the usual notation

$$(x)_k = x(x+1) \cdots (x+k-1)$$

we have

$$(\lambda)_{\mathbf{m}} = (\lambda)_{m_1} \left(\lambda - \frac{a}{2}\right)_{m_2} \cdots \left(\lambda - (r-1)\frac{a}{2}\right)_{m_r}. \tag{5.7}$$

The first statements are immediate from this.

The other statements are proved in the same way as in [17]: The invariance of the form $(\cdot|\cdot)_\lambda$ implies that M_0 is an invariant subspace of $\mathscr{P}^{(\lambda)}$; on the other hand, the action of $\mathfrak{p}^{\mathbb{C}}$ determines the form uniquely on M_0, which implies that M_0 is irreducible. Noting that the invariance of the form (5.6) is immediate from (5.5), the same argument can be applied inductively to each M_j/M_{j-1}. This finishes the proof.

The first statement of the next theorem was first proved by Wallach [25] in a somewhat different way.

5.4. THEOREM. *M_0 is unitarizable if and only if $\lambda > (r-1)\,a/2$ or $\lambda = ja/2$ with an integer $0 \leqslant j \leqslant r-1$. M_0 is finite dimensional if and only if λ is a non-positive integer. For $1 \leqslant j \leqslant q$, M_j/M_{j-1} is infinite dimensional. It is unitarizable if and only if $j = q$ and $(r-1)\,a/2 - \lambda$ is an integer; in this case it is isomorphic with $\mathscr{h}^{(2n_1/r-\lambda)}$.*

Proof. As well known, the unitarizability of M_0 amounts to the form $(\cdot|\cdot)_\lambda$ being (positive or negative) definite on M_0. By (5.4) this amounts to $(\lambda)_{\mathbf{m}}$ having identical sign for all $\mathbf{m}$ such that $\mathscr{h}_{\mathbf{m}} \subset M_0$, i.e., for all $\mathbf{m} \geqslant 0$ such that $(\lambda)_{\mathbf{m}} \neq 0$. Similarly, from (5.6) one sees that M_j/M_{j-1} is unitarizable if and only if the product $(\lambda)^{\sim}_{\mathbf{m}}$ obtained by developing (5.7) and canceling the linear factors that are equal to zero has the same sign for all $\mathbf{m}$ such that $\mathscr{h}_{\mathbf{m}} \subset M_j$ and $\mathscr{h}_{\mathbf{m}} \not\subset M_{j-1}$.

Consider λ fixed and consider the factors $(\lambda-(j-1)\,a/2)_{m_j}$ in (5.7) as functions of $\mathbf{m}$. We call those factors for which $\lambda-(j-1)\,a/2$ is a non-positive integer "eligible factors." There are q of these ($q=0$ is possible); if a is even they are just the last q factors, if a is odd they are either the factors of index r, $r-2, ..., r-2q+2$ or those of index $r-1$, $r-3, ...,$ $r-2q+1$, depending on the parity of r and the integrality of λ.

Suppose $q=0$. Then it is clear from (5.7) that $(\lambda)_{\mathbf{m}}$ has the same sign for all $\mathbf{m} \geqslant 0$ if and only if $\lambda > (r-1)\,a/2$. In the following we consider the case $q \geqslant 1$. The possible values of λ are then given by Theorem 5.3.

It is clear that if an eligible factor is zero, then so are all the preceding eligible factors. Let u be the index of the first eligible factor, that is, let u be the smallest integer between 1 and r such that $(u-1)\,a/2-\lambda$ is a non-negative integer. Clearly, the condition $\mathscr{h}_{\mathbf{m}} \subset M_0$ is then equivalent to $(\lambda-(u-1)\,a/2)_{m_u} \neq 0$, i.e., to

$$m_u \leqslant (u-1)\frac{a}{2}-\lambda. \tag{5.8}$$

All factors $(\lambda-(w-1)\,a/2)_{m_w}$ to the left of the u'th one (i.e., for $w<u$) are then positive for every $\mathbf{m}$. The u'th factor itself can certainly be 1 (taking $m_u=0$), and, in case $(u-1)\,a/2-\lambda>0$, it can also be negative (taking $m_u=1$), so in this case M_0 is not unitarizable. However, in case $(u-1)\,a/2-\lambda=0$ (and we are in this case exactly if $\lambda=ja/2$ with some $0 \leqslant j \leqslant r-1$), (5.8) gives $m_u=0$ and then we have also $m_w=0$ for all $w>u$. So all factors to the right of the u'th one are equal to 1, showing that M_0 is unitarizable.

As to the dimension of M_0, it is finite if and only if there are only finitely many $\mathbf{m} \geqslant 0$ satisfying (5.8), which means if and only if $u=1$, which, by definition of u, means if and only if λ is a non-positive integer.

Now let $1 \leqslant j \leqslant q$. The condition $\mathscr{h}_{\mathbf{m}} \subset M_j$, $\mathscr{h}_{\mathbf{m}} \not\subset M_{j-1}$ is equivalent to saying that $\mathbf{m}$ is such that exactly j of the eligible factors are zero. Since an eligible factor being zero implies that the preceding ones are zero as well, the condition is equivalent to the pair of inequalities

$$m_v > (v-1)\frac{a}{2} - \lambda \tag{5.9}$$

$$m_u \leqslant (u-1)\frac{a}{2} - \lambda, \tag{5.10}$$

where v is the index of the j'th and u the index of the $(j+1)$st eligible factor; in the case $j = q$, (5.10) is vacuous. Condition (5.9) is never vacuous, and its right hand side is a non-negative integer.

If (5.10) is not vacuous, its right hand side is larger by at least 1 than the right hand side of (5.9), and it follows that $m_u = 0$ and $m_u = 1$ are both possible. This gives values of $(\lambda)_{\mathbf{m}}^{\sim}$ of different signs, so M_j/M_{j-1} is not unitarizable.

If $j = q$, there is only the condition (5.9) on $\mathbf{m}$, and v equals r or $r-1$ depending on whether $(r-1)\,a/2 - \lambda$ is an integer or not. After canceling the zero linear terms, the v'th factor and the factors to the left of it will never change sign; in fact, $w \leqslant v$ implies $m_w \geqslant m_v$ and hence $m_w > (w-1)\,a/2 - \lambda$, which shows that changing m_w we only change everything by a positive factor. So, if $v = r$, this proves that M_q/M_{q-1} is unitarizable. On the other hand, if $v = r-1$, then $\lambda - (r-1)\,a/2 < 0$, and (5.9) still permits the values 0 and 1 for m_r. Therefore $(\lambda)_{\mathbf{m}}^{\sim}$ will not have constant sign as $\mathbf{m}$ changes, and M_q/M_{q-1} is not unitarizable.

To prove that M_q/M_{q-1}, when $q \geqslant 1$ and $(r-1)\,a/2 - \lambda$ is an integer, is isomorphic with $\mathscr{P}^{(2n_1/r - \lambda)}$, it suffices to prove that these two modules have the same highest weight. We consider the highest weights with respect to the maximal nilpotent subalgebra $\mathfrak{s} + \mathfrak{p}^+$ of $\mathfrak{g}^{\mathbb{C}}$; this corresponds to a somewhat unnatural ordering, but it makes it easy to use our computations from Section 2.

We write $\mu = (r-1)\,a/2 - \lambda + 1$; this is now a positive integer. We claim that Δ^μ is a highest weight vector in M_q/M_{q-1}. In fact, $j_X(z)$ for $X \in \mathfrak{k}^{\mathbb{C}}$ is just $\operatorname{tr} \operatorname{ad}_{\mathfrak{p}^+}(X)$, so it is zero for $X \in \mathfrak{s}$. We have $j_X(z) = 0$ for $X \in \mathfrak{p}^+$ as well, since P^+ acts by translations. $-X\Delta^\mu$ for $X \in \mathfrak{s}$ is the same as the $\pi_*(X)\,\Delta^\mu$ of Section 2, hence zero. Finally, for $X \in \mathfrak{p}^+$, $X\Delta^\mu$ is a derivative of the homogeneous polynomial Δ^μ, hence it has lower degree, and hence must be a sum of terms in $\mathscr{P}_{\mathbf{m}}$'s with $m_r < \mu$. Therefore $X\Delta^\mu$ is in M_{q-1}, i.e., it vanishes modulo M_{q-1}.

To determine the corresponding weight we apply (5.2) with $X \in \mathfrak{h}^{\mathbb{C}}$. By Theorem 2.1, $\pi_*(X)\,\Delta^\mu = -\mu \sum \gamma_j(X)$. To find $j_X(z) = \operatorname{tr} \operatorname{ad}_{\mathfrak{p}^+}(X)$ we use the decompositions (1.3), (1.4), and (1.5) and find that

$$j_X(z) = \left((r-1)\frac{a}{2} + 1 + \frac{b}{2}\right) \sum_1^r \gamma_j(X) = \frac{p}{2} \sum_1^r \gamma_j(X).$$

Hence, by (5.2), the maximal weight is

$$\left(-\mu-\frac{\lambda}{2}\right)\sum_1^r \gamma_j=\frac{1}{2}\left(\lambda-(r-1)\frac{a}{2}-1\right)\sum_1^r \gamma_j$$
$$=\frac{1}{2}\left(\lambda-\frac{2n_1}{r}\right)\sum_1^r \gamma_j.$$

The same arguments show that when $\not{p}^{(\lambda)}$ is irreducible the function 1 is a maximal weight vector, and the corresponding weight is $\frac{1}{2}d\sum\gamma_j$. This finishes the proof of the theorem.

Remarks added in proof. 1. The composition series of the modules $\not{p}^{(\lambda)}$ (i.e. essentially our Theorem 5.3) is described in slightly different terms in Proposition 2.4 of H. P. Jakobsen, Basic covariant differential operators on hermitian symmetric spaces, *Ann. Sci. Ecole Norm. Sup.* **18** (1985), 421–436. The question of unitarizability is treated in H. P. Jakobsen, Hermitian symmetric spaces and their unitary highest weight modules, *J. Funct. Analysis* **52** (1983), 385–412. 2. As it is easy to verify, our Theorem 3.6 proves for certain values of the parameters the conjecture (C5) in I. G. Macdonald, "Commuting Differential Operators and Zonal Spherical Functions," Algebraic groups Utrecht, 1986, *Springer Lecture Notes in Math.* **1271** (1987), 189–200.

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